THE EXTENSION OF BOUNDED HOLOMORPHIC FUNCTIONS FROM HYPERSURFACES IN A POLYCYLINDER

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Let S be an analytic subset of pure codimension 1 of a complex manifold G. Let f be a bounded, holomorphic function on S. Then, the following extension problem can be asked: *Does there exist a bounded holomorphic function F on G such that $F|S = f$?* Of course, in general such an extension is impossible. Even in simple cases, the answer is difficult, for instance in the case of the unit disc. Alexander [*1*] solved the problem for the polydisc under very restrictive assumptions on S^2. Stout [*6*] solved the second Cousin problem with bounds for the polydisc. His paper was the starting point for these investigations. Grauert and Lieb [*2*] and Ramírez de Arellano [*4*] solved the first Cousin problem with bounds if G is a strongly pseudoconvex open, bounded subset of $\mathbf{C}$ with C^∞-boundary.

Here, the problem will be solved under suitable assumptions, if G is a polycylinder, i.e., $G = G_1 \times \cdots \times G_n$, where each G_ν is an open, connected bounded subset of the complex plane $\mathbf{C}$. As Alexander has shown by a counterexample, the problem in general is unsolvable, i.e., it is unreasonably posed. Precisely, the result is as follows: Let h be a bounded, holomorphic function on G and suppose that $S = h^{-1}(0)$ is non empty and different from $G = G_1 \times \cdots \times G_n$. A holomorphic function $f: S \to \mathbf{C}$ is said to be *strictly bounded*, if and only if there exists a finite open covering $\{U_i\}_{i \in I}$ of $\bar{G}$, a bounded holomorphic function f_i on $U_i \cap G$ for each $i \in I$, and a bounded holomorphic function a_{ij} on $U_i \cap U_j \cap G$ for each $(i,j) \in I \times I$ with $U_i \cap U_j \neq \varnothing$ such that

$$f\big| U_i \cap S = f_i\big| U_i \cap S \qquad (i \in I)$$

$$f_i - f_j = a_{ij} h \qquad \text{on } U_i \cap U_j \cap G$$

if $U_i \cap U_j \neq \varnothing$. Theorem 2.8 asserts that a bounded holomorphic function $F: G \to \mathbf{C}$ with $F|S = f$ exists, if f is strictly bounded.

The proof uses a type of Grothendieck Lemma with bounds on $G_1 \times \cdots \times G_n$. The exact statement is given in Theorem 2.8.

1. *The Grothendieck Lemma with Bounds*

Notations. If $f: M \to \mathbf{C}$ is a function, define $\| f \|_M = \sup |f(M)|$. If M is a real or complex manifold, let $C^k(M)$ be the set of functions $f: M \to \mathbf{C}$ of class C^k, where $0 \leqq k \leqq \infty$ implies the usual meaning and $k = h$ means holomorphic. If $M \subseteq M_1 \times \cdots \times M_n$, let $C^k(M, k_1 \cdots k_n)$ be the set of all $f \in C^k(M)$, such that f is of class C^{k_ν} in the νth variable.

If M is a set, define $M^n = M \times \cdots \times M$ (n-times). Let $\mathbf{N}_0$ be the set of non negative integers. For $\mu \in \mathbf{N}_0^n$, define the differential operator

$$D_\mu = D_\mu^{\bar{z}} = \frac{\partial^{\mu_1 + \cdots + \mu_n}}{\partial \bar{z}_1^{\mu_1} \cdots d\bar{z}_n^{\mu_n}}.$$

If M is open in $\mathbf{C}^n$, if $f \in C^k(M)$ and if $0 \leqq p \leqq k$ and $0 \leqq \lambda \leqq n$, let $\| f \|_M^{(p,\lambda)}$ be the maximum of all $\| D_\mu f \|_M$ where $0 \leqq \mu_1 + \cdots + \mu_\lambda \leqq p$ and $\mu_{\lambda+1} = \cdots = \mu_n = 0$. Define $\| f \|_M^{(p)} = \| f \|_M^{(p,n)}$. Then

$$\| f \|_M^{(0)} = \| f \|_M = \| f \|_M^{(0,\lambda)}.$$

Define

$$C_{p,\lambda}^k(M) = \{ f \in C^k(M) \mid \| f \|_M^{(p,\lambda)} < \infty \},$$
$$C_p^k(M) = C_{p,n}^k(M).$$

If $K \subseteq H \subset \mathbf{C}$, let

$$\beta(K,H) = \inf\{ |z-w| \mid z \in K,\ w \in \mathbf{C} - H \}$$

be the boundary distance from K to H. If K is compact and $H \neq \mathbf{C}$, then $\infty > \beta(K,H) > 0$. If H is a bounded subset of $\mathbf{C}$, its diameter is given by $\operatorname{diam}(H) = \sup\{ |z-w| \mid (z,w) \in H \times H \}$. If H is a measurable subset of $\mathbf{C}$, let $m(H)$ be its Lebesgue measure.

Lemma 1.1. *Let M_1 be an open subset of $\mathbf{C}^n$ with $n \geqq 0$. Let M_2 be a complex manifold. For every subset H of $\mathbf{C}$ define $H' = M_1 \times H \times M_2$. Let G be a non empty, bounded, open subset of $\mathbf{C}$ with diameter Δ. Define $M = G'$.*
Then for every open set with $\varnothing \neq H \subset \bar{H} \subset G$, a linear map

$$\kappa = \kappa_H : C^\infty(M; \infty, \infty, h) \to C^\infty(\mathbf{C}^1; \infty, \infty, h)$$

exists such that for all non negative integers s

a) $$\| \kappa(f) \|_M^{(s,n)} \leqq 2\Delta \| f \|_M^{(s,n)} \leqq \infty,$$

b) $$\| \kappa(f)_{\bar{z}} \|_{\mathbf{C}}^{(s,n)} \leqq \| f \|_M^{(s,n)} \leqq \infty,$$

c) $$\kappa(f)_{\bar{z}} = f \quad \text{on } H',$$

d) *if K is compact, if H and $\bar{H}$ are open with*

$$\varnothing \neq K \subset H \subset \bar{H} \subset \tilde{H} \subset \bar{\tilde{H}} \subset G,$$

then

$$\| \kappa_H(f) - \kappa_{\tilde{H}}(f) \|_{K'}^{(s,n)} \leqq \frac{m(G-H)}{\beta(K,H)} \| f \|_M^{(s,n)}.$$

e) *If $\hat{M}_\nu$ is an open, non empty subset of M_ν, if H is open with $\varnothing \neq H \subset \bar{H} \subset G$ and if r are the natural restriction maps, then the following diagram is commutative*:

$$\begin{array}{ccc} C^\infty(M_1 \times G \times M_2; \infty, \infty, h) & \xrightarrow{\kappa_H} & C^\infty(M_1 \times \mathbf{C} \times M_2; \infty, \infty, h) \\ r \downarrow & & \downarrow r \\ C^\infty(\hat{M}_1 \times G \times \hat{M}_2; \infty, \infty, h) & \xrightarrow{\kappa_H} & C^\infty(\hat{M}_1 \times \mathbf{C} \times \hat{M}_2; \infty, \infty, h). \end{array}$$

Proof. Let H be open and bounded with $\bar{H} \subset G$. Take a function $\rho = \rho_H$ of class C^∞ on $\mathbf{C}$ with compact support in G such that $0 \leqq \rho \leqq 1$ and such that $\rho = 1$ in a neighborhood of $\bar{H}$. Take $f \in C^\infty(M; \infty, \infty, h)$. For $(x, z, w) \in \mathbf{C}' = M_1 \times \mathbf{C} \times M_2$, define

$$g(x,z,w) = \begin{cases} f(x,z,w)\rho(z) & \text{if } z \in G \\ 0 & \text{if } z \in \mathbf{C} - G. \end{cases}$$

Then $g \in C^\infty(\mathbf{C}'; \infty, \infty, h)$. For $(x, z, w) \in \mathbf{C}'$, define

$$\kappa(f) = \frac{1}{2\pi i} \int_G g(x,\zeta,w)\rho(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

If U contains G and has diameter $r \leqq \infty$, then

$$\kappa(f) = -\frac{1}{\pi} \int_0^r \int_0^{2\pi} g(x, te^{i\phi} + z, w) e^{-i\phi} \, d\phi dt,$$

on U'. If $U = G$, then $r = \Delta$. The expressions for $\kappa(f)$ show $\kappa(f) \in C^\infty(M; \infty, \infty, h)$. Obviously, κ is linear. Moreover, if $z \in G$, then

$$D_\mu^{\bar{x}} \kappa(f) = -\frac{1}{\pi} \int_0^{2\pi} \int_0^\Delta (D_\mu^{\bar{x}} g)(x, te^{i\phi} + z, w) e^{-i\phi} dt d\phi.$$

Hence

$$\left| D_\mu^{\bar{x}} \kappa(f) \right| \leqq 2\Delta \left\| D_\mu^{\bar{x}} g \right\|_{\mathbf{C}'} \leqq 2\Delta \left\| D_\mu^{\bar{x}} f \right\|_M$$

which implies a). Obviously e) is true. As is well known, $\kappa(f)_{\bar{z}} = g$ on $\mathbf{C}'$ (Hörmander [3, p. 3, Theorem 1.2.2]). Therefore, b) and c) are true.

Now, the assumptions of d) are made. Take $(x, z, w) \in K' = M_1 \times K \times M_2$: then

$$\begin{aligned}
&\left| D_\mu^{\bar{x}} \kappa_H(f)(x,z,w) - D_\mu^{\bar{x}} \kappa_{\tilde{H}}(f)(x,z,w) \right| \\
= &\left| \frac{1}{2\pi i} \int_G D_\mu^{\bar{x}} f(x,\zeta,w)(\rho_H(\zeta) - \rho_{\tilde{H}}(\zeta)) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z} \right| \\
\leqq &\frac{1}{2\pi} \int_{G-H} \left| (D_\mu^{\bar{x}} f)(x,\zeta,w) \right| \frac{i d\zeta \wedge d\bar{\zeta}}{|\zeta - z|} \\
\leqq &\frac{m(G-H)}{\beta(K,H)} \left\| D_\mu^{\bar{x}} f \right\|_M
\end{aligned}$$

which implies d). Q.E.D.

For $1 \leqq p \leqq n$, let $T(p,n)$ be the set of injective and increasing maps of $\{1, 2, \cdots, p\}$ into $\{1, \cdots, n\}$. If $\mu \in T(p,n)$ and $f_1, \cdots, f_n$ are differentiable functions, define

$$df_\mu = df_{\mu(s)} \wedge \cdots \wedge df_{\mu(p)}.$$

If G is open in $\mathbf{C}^n$, let $A^{p,q}(G)$ be the vector space of complex forms of bidegree (p,q) and of class C^∞ on G. Define

$$Z^{p,q}(G) = \{\omega \in A^{p,q}(G) \mid \bar{\partial}\omega = 0\}.$$

Let $z_1, \cdots, z_n$ be the coordinate functions on G. If $\omega \in A^{p,q}(G)$, then

$$\omega = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q,n)} \omega_{\mu\nu} dz_\mu \times d\bar{z}_\nu$$

where $\omega_{\mu\nu} \in C^\infty(G)$. Define

$$\begin{aligned}
\|\omega\|_G^{(s,\lambda)} &= \sup\{\|\omega_{\mu\nu}\|_G^{(s,\lambda)} \mid (\mu,\nu) \in T(p,n) \times T(q,n)\}, \\
\|\omega\|_G^{(s)} &= \|\omega\|_G^{(s,n)}, \\
\|\omega\|_G &= \|\omega\|_G^{(0)} = \|\omega\|_G^{(0,\lambda)}.
\end{aligned}$$

For $q \leqq \lambda \leqq n$, let $A_\lambda^{p,q}(G)$ be the set of $\omega \in A^{p,q}(G)$ with

$$\omega = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q,\lambda)} \omega_{\mu\nu} dz_\mu \wedge d\bar{z}_\nu.$$

Define $Z_\lambda^{p,q}(G) = A_\lambda^{p,q}(G) \cap Z^{p,q}(G)$. Now Lemma 1.1 extends:

Lemma 1.2. *Suppose $1 \leqq q \leqq n$. Let M_1 be open in $\mathbf{C}^{q-1}$ and M_2 be open in $\mathbf{C}^{n-q}$. If $H \subseteq \mathbf{C}$, define $H' = M_1 \times H \times M_2$. Let G be a non empty, bounded, open subset of $\mathbf{C}$ with diameter Δ. Define $M = G'$. Then, for every open set H with $\varnothing \neq H \subset \bar{H} \subset G$, a linear map*

$$K = K_H : Z_q^{p,q}(M) \to A_{q-1}^{p,q-1}(\mathbf{C}')$$

exists such that $K_H(\omega)$ is holomorphic in the variables $(z_{q+1}, \cdots, z_n)$ on M_2 if $\omega \in Z_q^{p,q}(M)$ and such that

a) $$\| K(\omega) \|_M \leqq 2\Delta \| \omega \|_M,$$

b) $$\bar{\partial} K(\omega) = \omega \quad \text{on } H',$$

c) *if C is compact, if H and $\tilde{H}$ are open with*

$$\varnothing \neq C \subset H \subset \bar{H} \subset \tilde{H} \subset \bar{\tilde{H}} \subset G$$

then

$$\| K_H(\omega) - K_{\tilde{H}}(\omega) \|_{C'} \leqq \frac{m(G - H)}{\beta(C, H)} \| \omega \|_M,$$

d) *if $\hat{M}_\nu$ is an open, non empty subset of M_ν for $\nu = 1, 2$, if H is open with $\varnothing \neq H \subset \bar{H} \subset G$ and if r denotes the natural restriction maps, then the following diagram is commutative:*

$$\begin{array}{ccc}
Z_q^{p,q}(M_1 \times G \times M_2) & \xrightarrow{K_H} & A_{q-1}^{p,q-1}(M_1 \times \mathbf{C} \times M_2) \\
r \big\downarrow & & \big\downarrow r \\
Z_q^{p,q}(\hat{M}_1 \times G \times \hat{M}_2) & \xrightarrow{K_H} & A_{q-1}^{p,q-1}(\hat{M}_1 \times \mathbf{C} \times \hat{M}_2).
\end{array}$$

Proof. If $\omega \in Z_q^{p,q}(M)$, then

$$\omega = \sum_{\mu \in T(p,n)} \omega_\mu dz_\mu \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$$

where $\omega_{\mu \bar{z}_\rho} = 0$ for $\rho = q + 1 \cdots, n$. Hence $\omega_\mu \in C^\infty(M; \infty, \infty, h)$. Define the linear map K by

$$K(\omega) = (-1)^{p+q-1} \sum_{\mu \in T(p,n)} \kappa_H(\omega_\mu) dz_\mu \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{q-1} .$$

Then $K_H(\omega) = K(\omega) \in A_{q-1}^{p,q-1}(\mathbf{C}')$ is holomorphic in the variables $(z_{q+1}, \cdots, z_n) \in M_2$. Lemma 1.1 implies a), c) and d) immediately. Moreover,

$$\begin{aligned}\bar{\partial}K(\omega) &= (-1)^{p+q-1} \sum_{\rho=1}^{n} \sum_{\mu \in T(p,n)} \kappa(\omega_\mu)_{\bar{z}_\rho} d\bar{z}_\rho \wedge dz_\mu \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{q-1} \\ &= \sum_{\mu \in T(p,n)} \kappa(\omega_\mu)_{\bar{z}_q} dz_\mu \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \\ &= \omega\end{aligned}$$

on H', because $\kappa(\omega_\mu)$ is holomorphic in $(z_{q+1}, \cdots, z_n)$. Q.E.D.

In the next lemma, the usual induction procedure will be used to prove Grothendieck's Lemma with bounds:

Lemma 1.3. *Let* $1 \leqq q \leqq \lambda \leqq n$. *Suppose that* $G = G_1 \times \cdots \times G_n$ *where each* G_ν *is a non empty, open subset of* $\mathbf{C}$. *Assume, that* $\Delta \geqq 1$ *exists with* $\Delta \geqq \operatorname{diam} G_\nu$ *for* $\nu = q, \cdots, n$. *Define*

$$G^\mu = G_1 \times \cdots \times G_\mu \qquad G(\mu) = G_{\mu+1} \times \cdots \times G_n.$$

If $H_\mu \subseteq \mathbf{C}$ *for* $\mu = q, \cdots, \lambda$, *define*

$$H^\mu = G^{q-1} \times H_q \times \cdots \times H_\mu \qquad H = H^\lambda.$$

Then, for every open set $H_q \times \cdots \times H_\lambda$ *with* $\varnothing \neq H_\mu \subset \bar{H}_\mu \subset G_\mu$ *for* $\mu = q, \cdots, \lambda$, *a linear map*

$$K = K_H : Z_\lambda^{p,q}(G) \to A_{\lambda-1}^{p,q-1}(H \times G(\lambda))$$

exists such that $K(\omega)$ *is holomorphic in the variables* $(z_{\lambda+1}, \cdots, z_n) \in G(\lambda)$ *if* $\omega \in Z_\lambda^{p,q}(G)$ *and such that*

a) $$\| K(\omega) \|_{H \times G(\lambda)} \leqq (6\Delta)^{\lambda-q+1} \| \omega \|_G^{(\lambda-q,\lambda-1)},$$

b) $$\bar{\partial}K(\omega) = \omega \qquad on\ H \times G(\lambda),$$

c) *if* C_μ *is compact, and if* H_μ *and* $\tilde{H}_\mu$ *are open with*

$$\varnothing \neq C_\mu \subset H_\mu \subset \bar{H}_\mu \subset \tilde{H}_\mu \subset \bar{\tilde{H}}_\mu \subset G_\mu$$

or $\mu = q, \cdots, \lambda$, *if* $C = C^\lambda \times G(\lambda)$, *if* $\tilde{H} = \tilde{H}^\lambda$, *if* $\beta = \min\{\beta(C_\mu, H_\mu) \mid \mu = q, \cdots, \lambda\}$ *and if* $\delta = \max\{m(G_\mu - H_\mu) \mid \mu = q, \cdots, \lambda\}$, *then*

$$\| K_H(\omega) - K_{\tilde{H}}(\omega) \|_C \leqq \frac{\delta}{\beta}(24\Delta)^{\lambda-q} \| \omega \|_G^{(\lambda-q,\lambda-1)},$$

d) *if* $\hat{G}_\nu$ *is an open, non empty subset of* G_ν *for* $\nu = \lambda+1, \cdots, n$, *if* H_μ *is open with* $\varnothing \neq H_\mu \subset \bar{H}_\mu \subset G_\mu$ *for* $\mu = q+1, \cdots, \lambda$, *then the diagram*

$$\begin{tikzcd}
Z_\lambda^{p,q}(G^\lambda \times G(\lambda)) \arrow[r, "K_H"] \arrow[d, "r"'] & A_{\lambda-1}^{p,q-1}(H \times G(\lambda)) \arrow[d, "r"] \\
Z_\lambda^{p,q}(G^\lambda \times \hat{G}(\lambda)) \arrow[r, "K_H"] & A_{\lambda-1}^{p,q-1}(H \times \hat{G}(\lambda))
\end{tikzcd}$$

is commutative, where r denotes the natural restriction maps.

Proof. The proof proceeds by induction for λ. Lemma 1.2 implies Lemma 1.3 for $q = \lambda$. Suppose that Lemma 1.3 is true for $\lambda - 1$ with $q < \lambda \leqq n$. Then, it shall be proved for λ as follows.

Take open, non empty sets $H_q, \cdots, H_\lambda$ with $\bar{H}_\mu \subset G_\mu$. Several linear maps shall be defined:

$$\pi\colon Z_\lambda^{p,q}(G) \to A_{\lambda-1}^{p,q-1}(G)$$

$$\psi\colon Z_\lambda^{p,q}(G) \to A_{\lambda-1}^{p,q}(G)$$

$$\chi\colon A_{\lambda-1}^{p,q-1}(G) \to A_{\lambda-1}^{p,q-1}(G)$$

$$\sigma\colon Z_\lambda^{p,q}(G) \to Z_{\lambda-1}^{p,q}(G^{\lambda-1} \times H_\lambda \times G(\lambda))$$

$$K\colon Z_\lambda^{p,q}(G) \to A_{\lambda-1}^{p,q-1}(H \times G(\lambda))$$

1. *Definition of π and ψ*: Take $\omega \in Z_\lambda^{p,q}(G)$, then $\omega = d\bar{z}_\lambda \wedge \alpha + \beta$ where $\alpha = \pi(\omega) \in A_{\lambda-1}^{p,q-1}(G)$ and $\beta = \psi(\omega) \in A_{\lambda-1}^{p,q}(G)$ are unique.

2. *Definition of χ*. Take $\alpha \in A_{\lambda-1}^{p,q-1}(G)$, then

$$\alpha = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q-1,\lambda-1)} \alpha_{\mu\nu} dz_\mu \wedge d\bar{z}_\nu$$

where $\alpha_{\mu\nu} \in C^\infty(G^{\lambda-1} \times G_\lambda \times G(\lambda))$. Define

$$\gamma = \chi(\alpha) = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q-1,\lambda-1)} \kappa_{H_\lambda}(\alpha_{\mu\nu}) dz_\mu \wedge d\bar{z}_\nu .$$

3. *Definition of σ*: If $\omega \in Z_\lambda^{p,q}(G)$ define

$$\sigma(\omega) = (\omega - \bar{\partial} \circ \chi \circ \pi(\omega)) \big| G^{\lambda-1} \times H_\lambda \times G(\lambda).$$

Then $\bar{\partial}\sigma(\omega) = \bar{\partial}\omega = 0$. Denote $\alpha = \pi(\omega)$ and $\beta = \psi(\omega)$. Then $\omega = d\bar{z}_\lambda \wedge \alpha + \beta$ with

$$\alpha = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q-1,\lambda-1)} \alpha_{\mu\nu} dz_\mu \wedge dz_\nu$$

$$\beta = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q,\lambda-1)} \beta_{\mu\nu} dz_\mu \wedge dz_\nu$$

$$\bar{\partial}\omega = -d\bar{z}_\lambda \wedge \sum_{\rho=1}^{n} \sum_{\mu\in T(p,n)} \sum_{\nu\in T(q-1,\lambda-1)} \alpha_{\mu\nu\bar{z}_\rho} d\bar{z}_\rho \wedge dz_\mu \wedge d\bar{z}_\nu$$

$$+ \sum_{\rho=1}^{n} d\bar{z}_\rho \wedge \sum_{\mu\in T(p,n)} \sum_{\nu\in T(q,\lambda-1)} \beta_{\mu\nu\bar{z}_\rho} dz_\mu \wedge d\bar{z}_\nu .$$

Therefore, $\alpha_{\mu\nu\bar{z}_\rho} = 0$ and $\beta_{\mu\nu\bar{z}_\rho} = 0$ if $\rho = \lambda+1,\cdots,n$. Hence $\alpha_{\mu\nu}$ and $\beta_{\mu\nu}$ are holomorphic in the variables $(z_{\lambda+1},\cdots,z_n)$ on $G(\lambda)$, which implies the same property for $\kappa_{H_\lambda}(\alpha_{\mu\nu})$. Set $\gamma = \chi(\alpha)$. Then

$$\begin{aligned}\bar{\partial}\gamma &= \sum_{\rho=1}^{n} \sum_{\mu\in T(p,n)} \sum_{\nu\in T(q-1,\lambda-1)} \kappa_{H_\lambda}(\alpha_{\mu\nu})_{\bar{z}_\rho} d\bar{z}_\rho \wedge dz_\mu \wedge d\bar{z}_\nu \\ &= d\bar{z}_\lambda \wedge \sum_{\mu\in T(p,n)} \sum_{\nu\in T(q-1,\lambda-1)} \alpha_{\mu\nu} dz_\mu \wedge d\bar{z}_\nu \\ &\quad + \sum_{\rho=1}^{\lambda-1} \sum_{\mu\in T(p,n)} \sum_{\nu\in T(q-1,\lambda-1)} \kappa_{H_\lambda}(\alpha_{\mu\nu})_{\bar{z}_\rho} d\bar{z}_\rho \wedge dz_\mu \wedge d\bar{z}_\nu \\ &= d\bar{z}_\lambda \wedge \alpha + \eta \quad \text{on} \quad G^{\lambda-1}\times H_\lambda \times G(\lambda),\end{aligned}$$

where $\eta \in A^{p,q}_{\lambda-1}(G)$. Hence

$$\sigma(\omega) = \omega - \bar{\partial}\gamma = \beta - \eta \in Z^{p,q}_{\lambda-1}(G^{\lambda-1}\times H_\lambda \times G(\lambda)).$$

4. *Definition of* $K = K_H$: Now, let τ be the operator K which exists by induction for $G_1,\cdots,G_{\lambda-1}$, H_λ, $G_{\lambda+1},\cdots,G_n$ as fixed domains and $H_q \subset \bar{H}_q \subset G_q,\cdots,H_{\lambda-1}\subset \bar{H}_{\lambda-1}\subset G_{\lambda-1}$ as subdomains. Observe that $H^{\lambda-1}\times H_\lambda \times G(\lambda) = H\times G(\lambda)$. Hence

$$\tau : Z^{p,q}_{\lambda-1}(G^{\lambda-1}\times H_\lambda\times G(\lambda)) \to A^{p,q-1}_{\lambda-2}(H\times G(\lambda))$$

where $A^{p,q-1}_{\lambda-2}(H\times G(\lambda))$ is contained in $A^{p,q-1}_{\lambda-1}(H\times G(\lambda))$. Define

$$K = K_H = \chi\circ\pi + \tau\circ\sigma : Z^{p,q}_{\lambda}(G) \to A^{p,q-1}_{\lambda-1}(H\times G(\lambda)).$$

Take $\omega\in Z^{p,q}_{\lambda}(G)$. Define $\alpha = \pi(\omega)$ and $\beta = \psi(\omega)$. Then $\omega = d\bar{z}_\lambda\wedge\alpha+\beta$. Then $\gamma = \chi(\alpha)\in A^{p,q-1}_{\lambda-1}(G)$ is holomorphic in $(z_{\lambda+1},\cdots,z_n)$ on $G(\lambda)$ as shown above. Moreover, $\tau(\sigma(\omega))$ is holomorphic in $(z_{\lambda+1},\cdots,z_n)$ on $G(\lambda)$ by induction. Hence $K(\omega)$ is holomorphic in $(z_{\lambda+1},\cdots,z_n)$ on $G(\lambda)$. Moreover, $\sigma(\omega) = \omega - \bar{\partial}\gamma$ and

$$\begin{aligned}\bar{\partial}K(\omega) &= \bar{\partial}\gamma + \bar{\partial}\tau(\omega-\bar{\partial}\gamma)\\ &= \bar{\partial}\gamma + \omega - \bar{\partial}\gamma = \omega\end{aligned}$$

on $H^{\lambda-1}\times H_\lambda\times G(\lambda) = H\times G(\lambda)$ which proves b).

Now a) shall be proved: Take $\omega \in Z_\lambda^{p,q}(G)$ and set again $\alpha = \pi(\omega)$, $\beta = \psi(\omega)$ and $\gamma = \chi(\alpha)$. Then $\omega = d\bar{z}_\lambda \wedge \alpha + \beta$ and $\sigma(\omega) = \omega - \bar{\partial}\gamma$ and

$$K(\omega) = \gamma + \tau(\sigma(\omega)).$$

Obviously,

$$\begin{aligned} \|\alpha\|_G^{(s,\rho)} &\leqq \|\omega\|_G^{(s,\rho)}, \\ \|\beta\|_G^{(s,p)} &\leqq \|\omega\|_G^{(s,p)}, \\ \|\gamma\|_G^{(s,\lambda-1)} &\leqq 2\Delta\|\alpha\|_G^{(s,\lambda-1} \leqq 2\Delta\|\omega\|_G^{(s,\lambda-1)}, \end{aligned}$$

by Lemma 1.1. Define

$$\eta = \sum_{\rho=1}^{\lambda-1} \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q-1,\lambda-1)} \kappa_{H_\lambda}(\alpha_{\mu\nu})_{\bar{z}_\rho} d\bar{z}_\rho \wedge dz_\mu \wedge d\bar{z}_\nu .$$

Then Lemma 1.1 a) implies

$$\|\eta\|_G^{(s,\lambda-1)} \leqq 2\Delta\|\alpha\|_G^{(s+1,\lambda-1)} \leqq 2\Delta\|\omega\|_G^{(s+1,\lambda-1)}.$$

Now, $\sigma(\omega) = \omega - \bar{\partial}\gamma = \beta - \eta$ on $\tilde{G} = G^{\lambda-1} \times H_\lambda \times G(\lambda)$ implies

$$\begin{aligned} \|\sigma(\omega)\|_{\tilde{G}}^{(s,\lambda-1)} &\leqq \|\beta\|_G^{(s,\lambda-1)} + \|\eta\|_G^{(s,\lambda-1)} \\ &\leqq \|\omega\|_G^{(s,\lambda-1)} + 2\Delta\|\omega\|_G^{(s+1,\lambda-1)} \\ &\leqq 3\Delta\|\omega\|_G^{(s+1,\lambda-1)} \end{aligned}$$

for any non negative integer s. By induction

$$\begin{aligned} \|\tau(\sigma(\omega))\|_{H\times G(\lambda)} &\leqq (6\Delta)^{\lambda-q}\|\sigma(\omega)\|_{\tilde{G}}^{(\lambda-q-1,\lambda+2)} \\ &\leqq \tfrac{1}{2}(6\Delta)^{\lambda-q+1}\|\omega\|_G^{(\lambda-q,\lambda-1)}. \end{aligned}$$

Hence

$$\begin{aligned} \|K(\omega)\|_{H\times G(\lambda)} &\leqq \|\gamma\|_G + \|\tau(\sigma(\omega))\|_{H\times G(\lambda)} \\ &\leqq 2\Delta\|\omega\|_G + \tfrac{1}{2}(6\Delta)^{\lambda-q+1}\|\omega\|_G^{(\lambda-q,\lambda-1)} \\ &\leqq (6\Delta)^{\lambda-q+1}\|\omega\|_G^{(\lambda-q,\lambda-1)} \end{aligned}$$

which proves a).

Now c) shall be proved: Suppose that the assumptions of c) are made. The constructions for $\tilde{H}_q, \cdots, \tilde{H}_\lambda$ are distinguished by $\sim$. Then $\tilde{\pi} = \pi$ and $\tilde{\psi} = \psi$. Take $\omega \in Z_\lambda^{p,q}(G)$. Then

$$K_H(\omega) - K_{\tilde{H}}(\omega) =$$

$$\chi(\pi(\omega)) - \tilde{\chi}(\pi(\omega)) + \tau(\sigma(\omega) - \tilde{\sigma}(\omega)) + \tau(\tilde{\sigma}(\omega)) - \tilde{\tau}(\tilde{\sigma}(\omega)).$$

Define $\alpha = \pi(\omega) = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q-1\ \lambda-1)} \alpha_{\mu\nu} dz_\mu \wedge d\bar{z}_\nu$. Then

$$\chi(\alpha) - \tilde{\chi}(\alpha) = \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q-1,\lambda-1)} (\kappa_{H_\lambda}(\alpha_{\mu\nu}) - \kappa_{\tilde{H}_\lambda}(\alpha_{\mu\nu})) dz_\mu \wedge d\bar{z}_\nu$$

where κ_{H_λ} and $\kappa_{\tilde{H}_\lambda}$ are formed for $G^{\lambda-1} \times H_\lambda \times G(\lambda)$ and $G^{\lambda-1} \times \tilde{H}_\lambda \times G(\lambda)$ respectively. Therefore Lemma 1.1 implies

$$\| \chi(\alpha) - \tilde{\chi}(\alpha) \|_C \leqq \frac{m(G_\lambda - H_\lambda)}{\beta(C_\lambda, H_\lambda)} \| \alpha \|_G \leqq \frac{\delta}{\beta} \| \omega \|_G .$$

Now

$$\tau = K_{H^{\lambda-1}}\colon Z^{p,q}_{\lambda-1}(G^{\lambda-1} \times H_\lambda \times G(\lambda)) \to A^{p,q}_{\lambda-1}(H^{\lambda-1} \times H_\lambda \times G(\lambda))$$
$$\tilde{\tau} = K_{\tilde{H}^{\lambda-1}}\colon Z^{p,q}_{\lambda-1}(G^{\lambda-1} \times \tilde{H}_\lambda \times G(\lambda)) \to A^{p,q}_{\lambda-1}(\tilde{H}^{\lambda-1} \times \tilde{H}_\lambda \times G(\lambda)).$$

Define

$$\tilde{\tau}_0 = K_{\tilde{H}^{\lambda-1}}\colon Z^{p,q}_{\lambda-1}\colon(G^{\lambda-1} \times H_\lambda \times G(\lambda)) \to A^{p,q}_{\lambda-1}(\tilde{H}^{\lambda-1} \times H_\lambda \times G(\lambda)).$$

The induction assumption d) implies $\tilde{\tau}(\tilde{\sigma}(\omega)) = \tilde{\tau}_0(\tilde{\sigma}(\omega))$. The induction assumption c) implies

$$\begin{aligned}
&\| \tilde{\tau}(\tilde{\sigma}(\omega)) - \tau(\tilde{\sigma}(\omega)) \|_C \\
\leqq\ & \| \tilde{\tau}_0(\tilde{\sigma}(\omega)) - \tau(\tilde{\sigma}(\omega)) \|_{C^{\lambda-1} \times H_\lambda \times G(\lambda)} \\
&\leqq \frac{\delta}{\beta}(24\Delta)^{\lambda-q-1} \| \tilde{\sigma}(\omega) \|^{(\lambda-q-1,\lambda-2)}_{G^{\lambda-1} \times \tilde{H}_\lambda \times G(\lambda)} \\
&\leqq \frac{\delta}{\beta}(24\Delta)^{\lambda-q-1} \| \tilde{\sigma}(\omega) \|^{(\lambda-q-1,\lambda-1)}_{G^{\lambda-1} \times \tilde{H}_\lambda \times G(\lambda)} \\
&\leqq \frac{\delta}{4\beta}(24\Delta)^{\lambda-q} \| \omega \|^{(\lambda-q,\lambda-1)}_G .
\end{aligned}$$

Now, $\sigma(\omega) = \beta - \eta$ on $G^{\lambda-1} \times H_\lambda \times G(\lambda)$ and $\tilde{\sigma}(\omega) = \beta - \tilde{\eta}$ on $G^{\lambda-1} \times \tilde{H}_\lambda \times G(\lambda)$ imply

$$\sigma(\omega) - \tilde{\sigma}(\omega) = \eta - \tilde{\eta} =$$

$$\sum_{\rho=1}^{\lambda-1} \sum_{\mu \in T(p,n)} \sum_{\nu \in T(q-1,\lambda-1)} (\kappa_{H_\lambda}(\alpha_{\mu\nu})_{\bar{z}_\rho} - \kappa_{\tilde{H}_\lambda}(\alpha_{\mu\nu})_{\bar{z}_\rho}) d\bar{z}_\rho \wedge dz_\mu \wedge d\bar{z}_\nu$$

on $G^{\lambda-1} \times H_\lambda \times G(\lambda)$ where $\alpha_{\mu\nu} \in C^\infty(G^{\lambda-1} \times G_\lambda \times G(\lambda); \infty, \infty, h)$. Let U be an open neighborhood of C_λ with $\bar{U} \subset H_\lambda$ such that $\beta(C_\lambda, H_\lambda) \leqq 2\beta(\bar{U}, H_\lambda)$. Lemma 1.1 d) implies

$$\begin{aligned} &\| \kappa_{H_\lambda}(\alpha_{\mu\nu}) - \kappa_{\tilde{H}_\lambda}(\alpha_{\mu\nu}) \|^{(s,\lambda-1)}_{G^{\lambda-1} \times U \times G(\lambda)} \\ &\qquad \leqq \frac{m(G_\lambda - H_\lambda)}{\beta(\bar{U}, H_\lambda)} \| \alpha_{\mu\nu} \|^{(s,\lambda-1)}_{G^{\lambda-1} \times G_\lambda \times G(\lambda)} \\ &\qquad \leqq \frac{2\delta}{\beta} \| \omega \|^{(s,\lambda-1)}_G . \end{aligned}$$

Hence

$$\| \kappa_{H_\lambda}(\alpha_{\mu\nu})_{\bar{z}_\rho} - \kappa_{\tilde{H}_\lambda}(\alpha_{\mu\nu})_{\bar{z}_\rho} \|^{(s,\lambda-1)}_{G^{\lambda-1} \times U \times G(\lambda)} \leqq \frac{2\delta}{\beta} \| \omega \|^{(s+1,\lambda-1)}_G$$

for $\rho = 1, \cdots, \lambda - 1$, which implies

$$\| \sigma(\omega) - \tilde{\sigma}(\omega) \|^{(s,\lambda-1)}_{G^{\lambda-1} \times U \times G(\lambda)} \leqq \frac{2\delta}{\beta} \| \omega \|^{(s+1,\lambda-1)}_G .$$

Define

$$\tau_0 = K_{H^{\lambda-1}} : Z^{p,q}_{\lambda-1}(G^{\lambda-1} \times U \times G(\lambda)) \to A^{p,q-1}_{\lambda-1}(H^{\lambda-1} \times U \times G(\lambda)).$$

The induction assumption d) implies $\tau_0(\sigma(\omega) - \tilde{\sigma}(\omega)) = \tau(\sigma(\omega) - \tilde{\sigma}(\omega))$. The induction assumption a) implies

$$\begin{aligned} &\| \tau(\sigma(\omega) - \tilde{\sigma}(\omega)) \|_C \\ &\qquad \leqq \| \tau_0(\sigma(\omega) - \tilde{\sigma}(\omega)) \|_{H^{\lambda-1} \times U \times G(\lambda)} \\ &\qquad \leqq (6\Delta)^{\lambda-q} \| \sigma(\omega) - \tilde{\sigma}(\omega) \|^{(\lambda-q-1,\lambda-1)}_{G^{\lambda-1} \times U \times G(\lambda)} \\ &\qquad \leqq \frac{\delta}{2\beta} (24\Delta)^{\lambda-q} \| \omega \|^{(\lambda-q,\lambda-1)}_G . \end{aligned}$$

Therefore

$$\begin{aligned} &\| K_H(\omega) - K_{\tilde{H}}(\omega) \|_C \\ &\leqq \| \chi(\alpha) - \tilde{\chi}(\alpha) \|_C + \| \tau(\sigma(\omega) - \tilde{\sigma}(\omega)) \|_C + \| \tau(\tilde{\sigma}(\omega)) - \tilde{\tau}(\tilde{\sigma}(\omega)) \|_C \\ &\leqq \frac{\delta}{\beta} \| \omega \|_G + \frac{\delta}{2\beta} (24\Delta)^{\lambda-q} \| \omega \|^{(\lambda-q,\lambda-1)}_G + \frac{\delta}{4\beta} (24\Delta)^{\lambda-q} \| \omega \|^{(\lambda-q,\lambda-1)}_G \\ &\leqq \frac{\delta}{\beta} (24\Delta)^{\lambda-q} \| \omega \|^{(\lambda-q,\lambda-1)}_G \end{aligned}$$

which proves c).

Now d) *shall be proved*: Suppose that the assumptions of d) are made. The constructions for $\hat{G}_{\lambda+1}, \cdots, \hat{G}_n$ are distinguished by $\hat{}$. Then $\hat{\pi} = \pi$ and $\hat{\psi} = \psi$. Take $\omega \in Z_\lambda^{p,q}(\hat{G})$. Then $\pi(\omega) = \hat{\pi}(\omega)$ and $\psi(\omega) = \hat{\psi}(\omega)$. Lemma 1.1 c) implies $\chi(\pi(\omega)) = \hat{\chi}(\pi(\omega)) = \hat{\chi}(\hat{\pi}(\omega))$. Hence

$$\hat{\sigma}(\omega) = \omega - \bar{\partial}\hat{\chi}(\hat{\pi}(\omega)) = \omega - \bar{\partial}\chi(\pi(\omega)) = \sigma(\omega).$$

Induction assumption d) implies $\hat{\tau}(\hat{\sigma}(\omega)) = \tau(\hat{\sigma}(\omega)) = \tau(\sigma(\omega))$. Therefore

$$\hat{K}_H(\omega) = \hat{\chi}(\hat{\pi}(\omega)) + \hat{\tau}(\hat{\sigma}(\omega)) = \chi(\pi(\omega)) + \tau(\sigma(\omega)) = K_H(\omega),$$

which proves d). Q.E.D.

For $\lambda = n$, the following theorem is obtained.

Theorem 1.4. *Let $1 \leqq q \leqq n$. Suppose that $G = G_1 \times \cdots \times G_n$, where each G_ν is a non empty, open subset of $\mathbf{C}$. Assume that $\Delta \geqq 1$ exists such that* $\operatorname{diam} G_\nu \leqq \Delta$ *for $\nu = q, \cdots, n$. Then, for every open set*

$$H = G_1 \times \cdots \times G_{q-1} \times H_q \times \cdots \times H_n \neq \varnothing$$

with $\bar{H}_\nu \subset G_\nu$ for $\nu = q, \cdots, n$ a linear map

$$K_H : Z^{p,q}(G) \to A^{p,q-1}(H)$$

exists such that

a) $$\| K_H(\omega) \|_H \leqq (6\Delta)^{n-q-1} \| \omega \|_G^{(n-q)}$$

b) $$\bar{\partial} K_H(\omega) = \omega \qquad \text{on } H,$$

c) *if C_μ is compact, if H_μ* and *$\tilde{H}_\mu$ are open and if*

$$\varnothing \neq C_\mu \subset H_\mu \subset \bar{H}_\mu \subset \tilde{H}_\mu \subset \bar{\tilde{H}}_\mu \subset G_\mu$$

for $\mu = q, \cdots, n$, if

$$C = G_1 \times \cdots \times G_{q-1} \times C_q \times \cdots \times C_n$$
$$\tilde{H} = G_1 \times \cdots \times G_{q-1} \times \tilde{H}_q \times \cdots \times \tilde{H}_n$$
$$H = G_1 \times \cdots \times G_{q-1} \times H_q \times \cdots \times H_n$$
$$\beta = \min\{\beta(C_\mu, H_\mu) \mid \mu = q, \cdots, n\}$$
$$\delta = \max\{m(G_\mu - H_\mu) \mid \mu = q, \cdots, n\}$$

then

$$\| K_H(\omega) - K_{\tilde{H}}(\omega) \|_C \leqq \frac{\delta}{\beta}(24\Delta)^{n-q} \| \omega \|_G^{(n-q)}.$$

Let G be an open subset of $\mathbf{C}^n$. Let $\tilde{A}^{p,q}(G)$ be the vector space of continuous forms of bidegree (p,q) on G. Take $\omega \in Z^{p,q}(G)$ with $q \geqq 1$, then $\psi \in \tilde{A}^{p,q-1}(G)$ is said to be a *strong solution* of the $\bar{\partial}$-problem of ω on G, if and only if a sequence $\{\psi_\nu\}_{\nu \in N}$ of forms $\psi_\nu \in A^{p,q-1}(G)$ exists such that the following conditions hold:

1. For every compact subset C of G there exists a number $\nu_C \in N$ such that $\bar{\partial}\psi_\nu = \omega$ on$C$ for all $\nu \geqq \nu_C$.

2. For every compact subset C of G

$$\| \psi - \psi_\nu \|_C \to 0 \qquad \text{for } \nu \to \infty .$$

If this is so write $\omega = \bar{\partial}!\psi$. Define the vector spaces

$$Z^{p,q}(G;s) = \{\omega \in Z^{p,q}(G) \mid \|\omega\|_G^{(s)} < \infty\}$$

$$\hat{A}^{p,q}(G) = \{\omega \in \tilde{A}^{p,q}(G) \mid \|\omega\|_G < \infty\}$$

$$A^{p,q}(G;s) = \{\omega \in A^{p,q}(G) \mid \|\omega\|_G^{(s)} < \infty\}.$$

Theorem 1.5. *Let $1 \leqq q \leqq n$. Suppose that G_ν is an open, non empty subset of* $\mathbf{C}$ *for $\nu = 1, \cdots, n$. Assume that $G_q, \cdots, G_n$ are bounded. Define $G = G_1 \times \cdots \times G_n$. Then a linear map*

$$L: Z^{p,q}(G; n-q) \to \hat{A}^{p,q-1}(G)$$

exists such that $\bar{\partial}!L(\omega) = \omega$ for all $\omega \in Z^{p,q}(G;n-q)$. If $\Delta \geqq 1$ is such that $\operatorname{diam} G_\nu \leqq \Delta$ *for $\nu = q, \cdots, n$ and if $\omega \in Z^{p,q}(G;n-q)$, then*

$$\| L(\omega) \|_G \leqq (6\Delta)^{n-q+1} \| \omega \|_G^{(n-q))}.$$

Proof. For each $\mu = q, \cdots, n$ choose a sequence $\{H_{\mu\nu}\}_{\nu \in N}$ of open, non empty sets $H_{\mu\nu}$ with

$$H_{\mu\nu} \subset \bar{H}_{\mu\nu} \subset H_{\mu\nu+1} \subset \bar{H}_{\mu\nu+1} \subset G_\mu$$

such that $G_\mu = \bigcup_{\nu \in N} H_\mu$

$$H^\nu = G_1 \times \cdots \times G_{q-1} \times H_{q\nu} \times \cdots \times H_{n\nu}.$$

Theorem 1.4 gives a linear map

$$Q_\nu = K_{H^{\nu+2}}: Z^{p,q}(G, n-q) \to A^{p,q-1}(H^{\nu+2}).$$

For each $\nu \in N$, take a C^∞-function ρ_ν on G with support in $H^{\nu+1}$ and with $\rho_\nu \mid \bar{H}^\nu = 1$ where $0 \leqq \rho_\nu \leqq 1$ on G. For $\omega \in Z^{p,q}(G, n-q)$ define $L_\nu(\omega) = \rho_\nu Q_\nu$ on $H^{\nu+2}$ and $L_\nu(\omega) = 0$ on $G - H^{\nu+2}$. Then $L_\nu(\omega) \in A^{p,q-1}(G)$.

Obviously, the map L_ν is linear. Take $\Delta \geqq 1$ with $\Delta \geqq \operatorname{diam} G_\nu$ for $\nu = q, \cdots, n$.

Let U be any open subset of G such that $\bar{U}$ is compact. Then a number $\nu_0(U) \in N$ exists such that $H^\nu \supset \bar{U}$ for $\nu \geqq \nu_0(U)$. Therefore

$$\| L_\nu(\omega) \|_U = \| Q_\nu(\omega) \|_U \leqq (6\Delta)^{n-q+1} \| \omega \|_G^{n-q} < \infty$$

for $\nu \geqq \nu_0(U)$ and $\omega \in Z^{p,q}(G; n-q)$. For $\nu \geqq \nu_0(U)$ define $\delta_\nu = \max\{m(G_\mu - H_{\mu\,\nu+2}) \mid \mu = q, \cdots, n\}$. Then $\delta_\nu \to 0$ for $\nu \to \infty$. Define

$$\beta(\nu) = \min\{\beta(\bar{U}, H_{\mu\nu+2}) \mid \mu = q, \cdots, n\}.$$

Then $\beta(\nu) \geqq \beta = \beta(\nu_0(U)) > 0$ for all $\nu \geqq \nu_0(U)$. If $\mu > \nu \geqq \nu_0(U)$, then

$$\begin{aligned} \| L_\nu(\omega) - L_\mu(\omega) \|_{\bar{U}} &= \| Q_\nu(\omega) - Q_\mu(\omega) \|_{\bar{U}} \\ &\leqq \frac{\delta_\nu}{\beta} (24\Delta)^{n-q} \| \omega \|_G^{(n-q)} . \end{aligned}$$

Hence $\{L_\nu(\omega)\}_{\nu \in N}$ is a Cauchy sequence on G and converges uniformly on every compact subset of G. Then

$$L(\omega) = \lim_{\nu \to \infty} L_\nu(\omega) \in \tilde{A}^{p,q-1}(G).$$

The map L is linear. Moreover

$$\| L(\omega) \|_{\bar{U}} \leqq (6\Delta)^{n-q+1} \| \omega \|_G^{(n-q)}$$

for every open, relative compact subset U of G. Hence

$$\| L(\omega) \|_G \leqq (6\Delta)^{n-q+1} \| \omega \|_G^{(n-q)} < \infty,$$

which implies $L(\omega) \in \hat{A}^{p,q-1}(G)$.

If C is compact in G, take an open neighborhood U of C with $\bar{U}$ compact and $\bar{U} \subset G$. If $\nu \geqq \nu_0(U)$, then

$$\bar{\partial} L_\nu(\omega) = \bar{\partial} Q_\nu(\omega) = \omega \text{ on } C$$

because $C \subseteq U \subseteq H^\nu$. Moreover,

$$\| L_\nu(\omega) - L(\omega) \|_{\bar{U}} \leqq \frac{\delta \nu}{\beta} (24\Delta)^{n-q} \| \omega \|_G^{(n-q)} .$$

Hence $\| L_\nu(\omega) - L(\omega) \|_C \to 0$ for $\nu \to \infty$. Consequently, $\bar{\partial}! L(\omega) = \omega$.

Q.E.D.

If $q = 1$, the situation can easily be improved:

Lemma 1.6. *Let $G \neq \emptyset$ be open in $\mathbf{C}^n$. Suppose that $\omega \in Z^{p,1}(G)$ and $\psi \in \tilde{A}^{p,0}(G)$ are given with $\omega = \bar{\partial}!\psi$. Then $\psi \in A^{p,0}(G)$ is of class C^∞ and $\omega = \bar{\partial}\psi$.*

Proof. Let $\{\psi_\nu\}_{\nu \in N}$ be a sequence of forms $\psi_\nu \in A^{p,0}(G)$ satisfying the conditions of the definition of $\omega = \bar{\partial}!\psi$. Let U be any open, relative compact subset of G with $\bar{U} \subseteq G$ such that $\omega = \bar{\partial}\phi$ on U where $\phi \in A^{p\,0}(U)$. A number $\nu_{\bar{u}} \in N$ exists such that $\bar{\partial}\psi_\nu = \omega = \bar{\partial}\phi$ on U for all $\nu \geqq \nu_{\bar{u}}$. Hence $\chi_\nu = \psi_\nu - \phi$ is a holomorphic form on U because $\bar{\partial}\chi_\nu = 0$ if $\nu \geqq \nu_{\bar{u}}$. Because $\psi_\nu \to \psi$ for $\nu \to \infty$ uniformly on U, it also follows that $\chi_\nu \to \chi = \psi - \phi$ for $\nu \to \infty$ uniformly on U. Hence χ is holomorphic on U, which implies that ψ is of class C^∞ on U and that $0 = \bar{\partial}\chi = \bar{\partial}\psi - \bar{\partial}\phi$ or $\bar{\partial}\psi = \omega$ on U. Because every point of G has such a neighborhood, $\psi \in A^{p,0}(G)$ and $\bar{\partial}\psi = \omega$ on G. Q.E.D.

Because $\bar{\partial}!L(\omega) = \omega$ Theorem 1.5 and Lemma 1.6 imply

Theorem 1.7. *Assume that $G = G_1 \times \cdots \times G_n \neq \emptyset$ is open and bounded in $\mathbf{C}^n$ with $G_\nu \subseteq \mathbf{C}$ for $\nu = 1, \cdots, n$. Let $\Delta \geqq 1$ be such that* $\operatorname{diam} G_\nu \leqq \Delta$ *for $\nu = 1, \cdots, n$. Then a linear map*

$$L: Z^{p,1}(G; n-1) \to A^{p,0}(G; 0)$$

exists such that $\bar{\partial}L(\omega) = \omega$ and

$$\| L(\omega) \|_G \leqq (6\Delta)^n \| \omega \|_G^{(n-1)} < \infty .$$

Observe, that $A^{0,0}(G,0)$ is the normed space of bounded complex valued C^∞-functions on G.

2. *Extension of Bounded Functions*

Let $G \neq \emptyset$ be open in the complex manifold M. If $U \neq \emptyset$ is open in the Hausdorff space $\bar{G}$, let $B_G(U)$ be the Banach space of bounded holomorphic functions on $U \cap G$. Observe that U is open in $\bar{G}$ and $U \cap G \neq \emptyset$ is open in M. If V is open in $\bar{G}$ with $\emptyset \neq V \subseteq U$, the restriction map $r_V^U : B_G(U) \to B_G(V)$ is defined. Then $B = B_G = \{B_G(U), r_V^U\}$ is the presheaf of bounded holomorphic functions on $\bar{G}$. It induces the *sheaf* $\mathfrak{B} = \mathfrak{B}(G)$ *of germs of bounded holomorphic functions* on $\bar{G}$. If $\mathfrak{O}$ is the sheaf of germs of holomorphic functions on M, then $\mathfrak{O} \,|\, G = \mathfrak{B} \,|\, G$, but $\mathfrak{O}_x \neq \mathfrak{B}_x$ may hold if $x \in \bar{G} - G$; of course $\mathfrak{B}_x \supseteq \mathfrak{O}_x$ for all $x \in \bar{G}$.

If $U \neq \emptyset$ is open in the Hausdorff space $\bar{G}$, let $\Gamma(U, \mathfrak{B})$ be the algebra of sections of $\mathfrak{B}$ over U. If V is open in $\bar{G}$ with $\emptyset \neq V \subseteq U$, the restriction map $r_V^U : \Gamma(U, \mathfrak{B}) \to \Gamma(V, \mathfrak{B})$ is defined. Then $\Gamma(\mathfrak{B}) = \{\Gamma(U, \mathfrak{B}), r_V^U\}$ is the canonical presheaf of $\mathfrak{B}$. Here, $\Gamma(U,\mathfrak{B})$ can be identified with an algebra of holomorphic functions on $U \cap G$, where $f \in \Gamma(U,\mathfrak{B})$ if and only if $f \mid (V \cap G) \in B_G(V)$ for every open set V in $\bar{G}$ whose closure $\bar{V}$ is compact and contained in U. Observe $B_G(U) \subseteq \Gamma(U,\mathfrak{B})$ and that B_G is a subpresheaf of $\Gamma(\mathfrak{B})$ under this identification. If $\bar{G}$ is compact, then $\Gamma(\bar{G},\mathfrak{B}) = B_G(\bar{G})$ is the algebra of bounded holomorphic functions on G.

Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a covering of $\bar{G}$ by non empty subsets U_i, which are open in the space $\bar{G}$. For $p \geqq 0$, define

$$I(p) = \{(i_0,\cdots,i_p) \mid U_{i_0} \cap \cdots \cap U_{i_p} \neq \varnothing\}.$$

Then $I = I(0)$. For $i = (i_0,\cdots,i_p) \in I(p)$ define $U_i = U_{i_0} \cap \cdots \cap U_{i_p}$. Let $\mathfrak{S}_{p+1}$ be the group of permutations on $(p+1)$ elements. If $\pi \in \mathfrak{S}_{p+1}$, define $\pi(i) = (i_{\pi(0)},\cdots,i_{\pi(p)})$ if $i = (i_0,\cdots,i_p)$. Then $\pi(I(p)) = I(p)$. A p-cochain f on $\mathfrak{U}$ in B is a family $f = \{f_i\}_{i \in I(p)}$ where $f_i \in B_G(U_i)$ and $f_{\pi(i)} = (\operatorname{sign}\pi)\cdot f_i$ for each $i \in I(p)$. The set $C^p(\mathfrak{U},B)$ of p-cochains forms a vector space over $\mathbf{C}$. A linear map $\delta: C^p(\mathfrak{U},B) \to C^{p+1}(\mathfrak{U},B)$ is defined by

$$(\delta f)_{i_0\ldots i_{p+1}} = \sum_{\nu=0}^{p+1} (-1)^\nu f_{i_0\ldots i_{\nu-1} i_{\nu+1}\ldots i_{p+1}}$$

where each term is restricted to $G \cap U_i$ with $i = (i_0,\cdots,i_p)$. Then $\delta \circ \delta = 0$. The vector space $Z^p(\mathfrak{U},B) = \{f \in C^p(\mathfrak{U},B) \mid \delta f = 0\}$ is the set of p-cocycles on $\mathfrak{U}$ in B and contains $\delta C^{p-1}(\mathfrak{U},B)$. For $p = 0$ define $H^0(\mathfrak{U},B) = Z^0(\mathfrak{U},B)$ and for $p \geqq 1$ define

$$H^p(\mathfrak{U},B) = Z^p(\mathfrak{U},B)/\delta C^{p-1}(\mathfrak{U},B)$$

as the pth cohomology group on $\mathfrak{U}$ in B. If $\mathfrak{B}$ is a refinement of $\mathfrak{U}$, then a linear map $r^{\mathfrak{U}}_{\mathfrak{B}}: H^p(\mathfrak{U},B) \to H^p(\mathfrak{B},B)$ is defined such that $r^{\mathfrak{B}}_{\mathfrak{W}} \circ r^{\mathfrak{U}}_{\mathfrak{B}} = r^{\mathfrak{U}}_{\mathfrak{W}}$ if $\mathfrak{W}$ is a refinement of $\mathfrak{B}$. Hence the direct limit of $\{H^p(\mathfrak{U},B), r^{\mathfrak{U}}_{\mathfrak{B}}\}$ exists and is $\{H^p(\bar{G},B), r^{\mathfrak{U}}\}$ where $r^{\mathfrak{U}}: H^p(\mathfrak{U},B) \to H^p(\bar{G},B)$ is the residual map and where $H^p(\bar{G},B)$ is the pth cohomology group in $\bar{G}$ with coefficients in B. Observe $H^0(\mathfrak{U},B) = H^0(\bar{G},B) = H^0(\bar{G},B) = \Gamma(\bar{G},\mathfrak{B})$ with $r^{\mathfrak{U}}_{\mathfrak{W}}$ and $r^{\mathfrak{U}}$ being the identity.

If in these definitions the presheaf $B = B_G$ is replaced by the presheaf $\Gamma(\mathfrak{B})$, the vector spaces

$$\begin{aligned} C^p(\mathfrak{U},\Gamma(\mathfrak{B})) &= C^p(\mathfrak{U},\mathfrak{B}) \\ Z^p(\mathfrak{U},\Gamma(\mathfrak{B})) &= Z^p(\mathfrak{U},\mathfrak{B}) \\ H^p(\mathfrak{U},\Gamma(\mathfrak{B})) &= H^p(\mathfrak{U},\mathfrak{B}) \\ H^p(\bar{G},\Gamma(\mathfrak{B})) &= H^p(\bar{G},\mathfrak{B}) \end{aligned}$$

are defined. The inclusion $B \subseteq \Gamma(\mathfrak{B})$ of presheaves defines an isomorphism $H^p(\bar{G}, B) \approx H^p(\bar{G}, \mathfrak{B})$, because $\bar{G}$ is paracompact.

Suppose that $\bar{G}$ is compact. Then every open covering on $\bar{G}$ has a finite refinement. Therefore, the previous considerations can be restricted to finite, open coverings. If $\mathfrak{U}$ is such a finite, open covering of $\bar{G}$, and if $f \in C^p(\mathfrak{U}, B)$ define

$$\| f \| = \max \{ \| f_i \| \mid i \in I(p) \}.$$

Then $\| \ \|$ is a norm on $C^p(\mathfrak{U}, B)$ and $C^p(\mathfrak{U}, B)$ is a Banach space with this norm. Moreover, $\| \delta f | \leqq (p+1) \| f \|$. Hence δ is continuous, and $Z^p(\mathfrak{U}, B)$ is again a Banach space with this norm.

Theorem 2.1. *Let $G = G_1 \times \cdots \times G_n \neq \emptyset$ be open and bounded in $\mathbf{C}^n$ with $G_\nu \subset \mathbf{C}$ for $\nu = 1, \cdots, n$. Let B be the presheaf of germs of bounded holomorphic functions on $\bar{G}$. Let $\mathfrak{U}$ be a finite open covering of the compact Hausdorff space $\bar{G}$. Then a continuous linear map*

$$\sigma : Z^1(\mathfrak{U}, B) \to C^0(\mathfrak{U}, B)$$

exists such that that $\delta \circ \sigma$ is the identity on $Z^1(\mathfrak{U}, B)$. Especially, $H^1(\mathfrak{U}, B) = 0$.

Proof. Take a partition $\{\rho_i\}_{i \in I}$ of unity by non negative C^∞-functions ρ_i on $\mathbf{C}^n$, such that supp ρ_i is compact and $\bar{G} \cap \operatorname{supp} \rho_i \subseteq U_i$ for each $i \in I$ and such that $\sum_{i \in I} \rho_i = 1$ on $\bar{G}$. Now a continuous linear map $\alpha : Z^1(\mathfrak{U}, B) \to Z^{0,1}(G; n-1)$ will be defined.

Take a number $C > 1$ such that

$$\| \rho_{i\bar{z}_\lambda} \|_{U_i}^{(s)} \leqq C$$

for all $i \in I$, all $\lambda = 1, \cdots, n$, and all $s = 0, \cdots, n$. Take

$$f = \{f_{ij}\}_{(i,j) \in I(1)}$$

in $Z^1(\mathfrak{U}, B)$. Then $f_{ij} = -f_{ij}$ on $U_i \cap U_j \cap G$ if $(i,j) \in I(1)$ and $f_{ij} + f_{jk} + f_{ki} = 0$ on $U_i \cap U_j \cap U_k \cap G$ if $(i,j,k) \in I(2)$. Define $f_{ij}(x) = 0$ if $x \in G - U_i \cap U_j$. Define $g_i = \sum_{k \in I} \rho_k f_{ki}$ on $U_i \cap G$. Then g_i is a function of class C^∞ on U_i with $\| g_i \|_{U_i \cap G} \leqq \| f \|$. Moreover,

$$g_j - g_i = \sum_{k \in I} \rho_k (f_{kj} + f_{ik}) = \sum_{k \in I} \rho_k f_{ij} = f_{ij}$$

on $U_i \cap U_j \cap G$ if $(i,j) \in I(1)$. Because f_{ij} is holomorphic $\bar{\partial} g_j = \bar{\partial} g_i$ on $U_i \cap U_j \cap G$. Hence $\omega \in Z^{0,1}(G)$ exists uniquely, such that $\omega \mid U_i \cap G = \bar{\partial} g_i$. Then

$$\omega = \sum_{\lambda=1}^{n} \omega_\lambda d\bar{z}_\lambda$$

on G where

$$\omega_\lambda = \sum_{k \in I} \rho_{k\bar{z}_\lambda} f_{ki} \text{ on } U_i.$$

If $\mu \in N_0^n$, then

$$D_\mu^{\bar{z}} \omega_\lambda = \sum_{k \in I} (D_\mu^{\bar{z}} \rho_{k\bar{z}_\lambda}) f_{ki},$$

which implies

$$\| \omega \|_G^{(n-1)} \leqq \max_{i \in I} \| \omega \|_{U_i \cap G}^{(n-1)} \leqq C \| f \| < \infty.$$

Define $\alpha(f) = \omega$. Then α is a linear map into $Z^{0,1}(G; n-1)$ with $\| \alpha(f) \|_G^{(n-1)} \leqq C \| f \|$. Hence α is continuous.

According to Theorem 1.7, a continuous, linear map $L: Z^{0,1}(G; n-1) \to A^{0,0}(G,0)$ with $\bar{\partial} L(\phi) = \phi$ exists, where $A^{0,0}(G,0)$ is the normed space of bounded C^∞-functions on G. Define $\psi = L(\alpha(f))$ on G. Define $h_i = g_i - \psi$ on $U_i \cap G$ for each $i \in I$. Then $\bar{\partial} h_i = \bar{\partial} g_i - \bar{\partial} \psi = \omega - \omega = 0$ on $U_i \cap G$. Hence h_i is holomorphic on $U_i \cap G$. If $\Delta \geqq 1$, such that $\operatorname{diam} G_\nu \leqq \Delta$ for $\nu = 1, \cdots, n$. Then

$$\| h_i \|_{U_i \cap G} \leqq C(6\Delta)^n \| f \| + \| f \| < \infty.$$

Hence, $h = \{h_i\}_{i \in I} \in C^0(\mathfrak{U}, B)$. Define $h = \sigma(f)$. Obviously, σ is linear. Because

$$\| \sigma(f) \| \leqq 2C(6\Delta)^n \| f \|$$

σ is continuous. If $(i,j) \in I(1)$, then

$$(\delta\sigma(f))_{ij} = (\delta h)_{ij} = h_j - h_i = g_j - g_i = f_{ij}.$$

Therefore, $(\delta \circ \sigma)(f) = f$. Q.E.D.

Theorem 2.2. *Let $G = G_1 \times \cdots \times G_n \neq \emptyset$ be open and bounded in $\mathbf{C}^n$ with $G_\nu \subset \mathbf{C}$ for $\nu = 1, \cdots, n$. Let $\mathfrak{B}$ be the sheaf of germs of bounded holomorphic functions on $\bar{G}$. Then*

$$H^1(\bar{G}, \mathfrak{B}) = 0.$$

Proof. Take $h \in H^1(\bar{G}, B) \approx H^1(\bar{G}, \mathfrak{B})$. Then a finite, open covering $\mathfrak{U}$ of the compact Hausdorff space $\bar{G}$ exists such that $f \in H^1(\mathfrak{U}, B)$ with $h = r^{\mathfrak{U}}(f)$ exists. By Theorem 2.1, $f = 0$, hence $h = 0$. Q.E.D.

Let $G \neq \emptyset$ be open in the complex manifold M. Let $\mathfrak{B}$ be the sheaf of germs of bounded holomorphic functions on $\bar{G}$. If U is open in $\bar{G}$, if $x \in U$ and if $h \in B_G(U)$ (or $h \in \Gamma(U,\mathfrak{B})$), then h defines a germ $h_x \in \mathfrak{B}_x$. Define $h\mathfrak{B} \mid U = \bigcup_{x \in U} \{f h_x \mid f \in \mathfrak{B}_x\}$. Then $h\mathfrak{B} \mid U$ is a subsheaf of $\mathfrak{B} \mid U$ generated by h. A subsheaf $\mathfrak{F}$ of $\mathfrak{B}$ is said to be a principal ideal sheaf of $\mathfrak{B}$ if and only if for every $x \in \bar{G}$ an open neighborhood U of x in $\bar{G}$ and an element $h \in B_G(U)$ exist such that $\mathfrak{F} \mid U = h\mathfrak{B} \mid U$. If it is possible to choose $U = \bar{G}$, then $\mathfrak{F}$ is called a globally principal ideal sheaf in $\mathfrak{B}$.

For any sheaf $\mathfrak{F}$ on $\bar{G}$, and any open subset U of the space $\bar{G}$, denote by $\Gamma(U,\mathfrak{F})$ the set of continuous sections in $\mathfrak{F}$ over U. If $\mathfrak{F}$ is a subsheaf of $\mathfrak{B}$, then $\Gamma(U,\mathfrak{F}) = \{f \in \Gamma(U,\mathfrak{B}) \mid f_x \in \mathfrak{F}_x \text{ for all } x \in U\}$.

Lemma 2.3. *Let $G \neq \emptyset$ be an open subset of a complex manifold M. Let $\mathfrak{B}$ be the sheaf of germs of bounded holomorphic functions on $\bar{G}$. Let $U \neq \emptyset$ be open in the space $\bar{G}$. Let $\mathfrak{F}$ be a subsheaf of $\mathfrak{B}$. Suppose that $h \in \Gamma(U,\mathfrak{B})$ exists such that $\mathfrak{F} \mid U = h\mathfrak{B} \mid U$. Then for every $s \in \Gamma(U,\mathfrak{F})$, a function $g \in \Gamma(U,\mathfrak{B})$ exists such that $s = gh$. Observe: If V is any open subset of G whose closure $\bar{V}$ is compact and contained in U, then g is bounded on $V \cap G$, i.e., $g \mid (V \cap G) \in B_G(V)$.*

Proof. Pick $s \in \Gamma(U,\mathfrak{F})$. Then $s \in \Gamma(U,\mathfrak{B})$ with $s_x \in \mathfrak{F}_x$ for each $x \in U$. Hence for every $x \in U$ a germ $f(x) \in \mathfrak{B}_x$ exists such that $s_x = f(x) \cdot h_x$. An open neighborhood $W(x)$ of x in U exists such that $g^x \cdot h = s$ on $W(x) \cap G$ with $g^x \in B_G(W(x))$. If h is identically zero on some component of $W(x) \cap G$, then s is identically zero on this component; therefore it can be assumed that $g^x = 1$ on this component. Then $g^x = g^y$ on $W(x) \cap W(y) \cap G$ if this intersection is not empty. Hence one and only one holomorphic function g exists on $U \cap G$ such that $g \mid W(x) \cap G = g^x$. Hence $g \cdot h = s$ on $U \cap G$. Let V be open in G such that $\bar{V}$ is compact and contained in U. There, finitely many points $x_1, \cdots, x_p$ exist in U such that $\bar{V} \subseteq W(x_1) \cup \cdots \cup W(x_p)$. Hence

$$\| g \|_{V \cap G} \leqq \max_{\nu = 1, \dots p} \| g^{x_\nu} \|_{W(x_\nu) \cap G} < \infty .$$

Therefore $g \mid V \cap G \in B_G(V)$. (Observe, that $\bar{V} \cap G$ may not be compact!) Q.E.D.

Now, Stout's result [6] can be formulated easily:

Theorem 2.4 (Stout). Let $G = G_1 \times \cdots \times G_n$, where each $G_\nu = \{z \in \mathbf{C} \mid |z| < 1\}$ is the unit disc. Let $\mathfrak{B}$ be the sheaf of germs of bounded holomorphic functions on $\bar{G}$. Then any principal ideal sheaf $\mathfrak{F}$ in $\mathfrak{B}$ is globally principal.

Proof. Finite, open coverings $\{U_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ of $\bar{G}$ with $\bar{V} \subset U_i$ exist such that $\mathfrak{F} \mid U_i = h_i \mathfrak{B} \mid U_i$ with $h_i \in B_G(U_i)$. Here, $\bar{V}_i$ is compact. Lemma 2.3 implies the existence of bounded holomorphic functions g_{ij} on $V_i \cap V_j \cap G$ if $V_i \cap V_j \neq \emptyset$ such that $h_i = g_{ij} h_j$ on $V_i \cap V_j \cap G$ where $g_{ij} = 1$ on every component of $V_i \cap V_j \cap G$ where $h_j \equiv 0$ and hence $h_i \equiv 0$. Then $g_{ji} = g_{ij}^{-1}$; especially, $g_{ij} \neq 0$ on $V_i \cap V_j \cap G$. Hence $\{h_i, V_i\}_{i \in I}$ is a Cousin II distribution in the sense of Stout [6]. Therefore, $h \in B_G(\bar{G})$ exists such that $h = u_i h_i$ and $h_i = v_i h$ on $V_i \cap G$ where u_i and v_i are bounded holomorphic functions on V_i. Clearly

$$\mathfrak{F} \mid V_i = h_i \mathfrak{B} \mid V_i = h v_i \mathfrak{B} \mid V_i \leqq h \mathfrak{B} \mid V_i,$$

$$h \mathfrak{B} \mid V_i = u_i h_i \mathfrak{B} \mid V = u_i \mathfrak{F} \mid V_i \leqq \mathfrak{J} \mid V_i.$$

Hence $\mathfrak{F} \mid V_i = h \mathfrak{B} \mid V_i$. Since $\{V_i\}_{i \in I}$ is a covering, it follows that $\mathfrak{F} = h \mathfrak{B}$.
Q.E.D.

Theorem 2.5. *Let $G = G_1 \times \cdots \times G_n \neq \emptyset$ be an open, bounded and connected subset of $\mathbf{C}^n$ with $G_\nu \subset \mathbf{C}$ for $\nu = 1, \cdots, n$. Let $\mathfrak{B}$ be the sheaf of bounded holomorphic functions on $\bar{G}$. Let $\mathfrak{F}$ be a globally principal ideal sheaf in $\mathfrak{B}$. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a finite, open covering of $\bar{G}$. Then*

$$H^1(\bar{G}, \mathfrak{F}) = 0.$$

Proof. If $\mathfrak{F}$ is the zero ideal, the statement is true. Suppose $\mathfrak{F}$ is not the zero ideal. Take $h \in B_G(\bar{G})$ such that $\mathfrak{F} = h\mathfrak{B}$. Then h is not identically zero on G. Because G is connected, $h_x \neq 0$ for each $x \in \bar{G}$ and h_x is not a zero-divisor in $\mathfrak{B}_x$. If $g \in \mathfrak{F}_x$, then $g = u \cdot h_x$ with $u \in \mathfrak{B}_x$. Because h_x is not a zero divisor, u is unique. Define $u = \lambda(g)$. Obviously, $\lambda: \mathfrak{J} \to \mathfrak{B}$ is linear and bijective. It remains to be shown that λ is continuous. Pick $g_0 \in \mathfrak{J}_x$ and let W be an open neighborhood of $u_0 = \lambda(g_0) \in \mathfrak{B}_x$. An open neighborhood U of x in $\bar{G}$ and $s \in B_G(U)$ and $t \in B_G(U)$ exist such that $s_x = u_0$ and $s_z \in W$ for all $z \in U$. Moreover, $t_x = g_0$ and $t_z \in \mathfrak{J}_z$ for all $z \in U$. Then $t_x = s_x h_x$. Hence $t = sh$ on $V \cap G$ for a neighborhood V of x in $\bar{G}$ with $V \subseteq U$. Hence $t_z = s_z h_z$ for $z \in V$, which implies $\lambda(t_z) = s_z \in W$ for all $z \in V$. Now, $y = \{t_z \mid z \in V\}$ is an open neighborhood of $g_0 = t_x$ with $\lambda(y) \subseteq W$. Hence λ is continuous. Therefore $\lambda: \mathfrak{J} \to \mathfrak{B}$ is a sheaf isomorphism (for $\mathfrak{B}$-modules). Hence isomorphisms

$$\lambda^*: H^1(\mathfrak{U}, \mathfrak{J}) \to H^1(\mathfrak{U}, \mathfrak{B})$$

$$\lambda^*: H^1(\bar{G}, \mathfrak{J}) \to H^1(\bar{G}, \mathfrak{B})$$

are induced. Now, Theorem 2.1 and Theorem 2.2 complete the proof. Q.E.D.

Let $G \neq \emptyset$ be an open subset of the complex manifold M. Let $\mathfrak{B}$ be the sheaf of germs of bounded holomorphic functions on M. Let $\mathfrak{J}$ be a principal ideal sheaf in $\mathfrak{B}$. Let $\mathfrak{Q}$ be the quotient sheaf. This defines the exact sequence.

$$0 \to \mathfrak{J} \to \mathfrak{B} \xrightarrow{\mu} \mathfrak{Q} \to 0$$

where μ is the residual map. Observe that $\mathfrak{J}$, $\mathfrak{B}$ and $\mathfrak{Q}$ are sheaves on $\bar{G}$, not only on G. Then

$$\operatorname{supp}\mathfrak{Q} = \{x \in \bar{G} \,|\, \mathfrak{Q}_x \neq 0\} = \{x \in \bar{G} \,|\, \mathfrak{J}_x \neq \mathfrak{B}_x\}.$$

The intersection $S(\mathfrak{J}) = G \cap \operatorname{supp}\mathfrak{Q}$ is analytic. If $\mathfrak{J} \neq \mathfrak{B}$, and if $\mathfrak{J}_x \neq 0$ for at least one point in each component of G, then $S(\mathfrak{J})$ has pure codimension 1. Observe that $(\operatorname{supp}\mathfrak{Q}, \mathfrak{Q})$ is a ringed space. $\mathfrak{Q}_x$ may have zero divisors. For each open subset U of the space $\bar{G}$ with $V = U \cap \operatorname{supp}\mathfrak{Q}$ $\neq \emptyset$, $\Gamma(V,\mathfrak{Q}) = \Gamma(U,\mathfrak{Q})$ is a ring. Its element are, so to speak, the "holomorphic functions" on U in $(\operatorname{supp}\mathfrak{Q},\mathfrak{Q})$. However, in general, this ring cannot be identified with a ring of functions on V.

If N is any analytic subset of G, and if $V \neq \emptyset$ is open in the space N, let $A(V;N)$ be the ring of holomorphic functions on V in the sense of Serre. If U is open in the space $\bar{G}$ and if $V = U \cap \operatorname{supp}\mathfrak{Q} \neq \emptyset$, then a ring homomorphism

$$v\colon \Gamma(V,\mathfrak{Q}) \to A(S(\mathfrak{J}) \cap V, S(\mathfrak{J}))$$

shall be defined: Take $s \in \Gamma(V,\mathfrak{Q})$. For every $x \in V$ an open neighborhood W_x of x in the space $\bar{G}$ and $f^x \in B_G(W_x)$ exist such that $\mu(f_z^x) = s(z)$ for all $z \in W_x$, where $s(z) = 0$ for $z \in U - V$. If $W_x \cap W_y \neq \emptyset$, then $f_z^x - f_z^y \in \mathfrak{J}_z$ for all $z \in W_x \cap W_y$. Hence the function

$$v(s)\colon S(\mathfrak{J}) \cap V \to \mathbf{C}$$

is well defined by $v(s)(z) = f^y(z)$ if $z \in W_x \cap S(\mathfrak{J})$. Obviously, $v(s) \in A(S(\mathfrak{J}) \cap V, S(\mathfrak{J}))$. Obviously, v is a ring-homomorphism, but, in general, v is not injective or surjective. $v(s)(z)$ is said to be *the value of s at z*. If $x \in V$, then $f^x \,|\, W_x \cap S(\mathfrak{J}) = v(s) \,|\, W_x \cap S(\mathfrak{J})$. Hence, for each point $x \in V$ a neighborhood $W = W_x$ exists such that $v(s)$ is bounded on $W_x \cap S(\mathfrak{J})$. Hence, if $\bar{G}$ is compact, and $s \in \Gamma(\bar{G},\mathfrak{Q})$, then $v(s)$ is bounded on $S(\mathfrak{J})$, because $\operatorname{supp}\mathfrak{Q}$ is compact and can be covered by finitely many neighborhoods W_x. In general, not every bounded holomorphic function on $S(\mathfrak{J})$ can be obtained this way. (See the example of Alexander at the end of this paper.)

A holomorphic function $f \in A(G, S(\mathfrak{J}))$ is said to be *strictly bounded* for $\mathfrak{J}$ if there exists an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of the space $\bar{G}$ and a family $\{f_i\}_{i \in I}$ of functions $f_i \in B_G(U_i)$ such that:

1) If $(i,j) \in I(1)$ and if $z \in U_i \cap U_j$, then $(f_i - f_j)_z \in \mathfrak{J}_z$.

2) If $i \in I$ with $U_i \cap S(\mathfrak{J}) \neq \varnothing$ then $f_i \,|\, U_i \cap S(\mathfrak{J}) = f \,|\, U_i \cap S(\mathfrak{J})$.

If $\bar{G}$ is compact, finitely many U_i cover $\bar{G}$ and f is bounded.

Lemma 2.6. *The function $f \in A(G, S(\mathfrak{J}))$ is strictly bounded for $\mathfrak{J}$, if and only if $f = v(s)$ for some $s \in \Gamma(\operatorname{supp}\mathfrak{Q}, \mathfrak{Q})$.*

Proof. a) If f is strictly bounded, define s by $s(z) = \mu(f_{iz})$ for $z \in U_i$. Because of 1), $s(z)$ does not depend on i. Hence s is well defined: $s: \bar{G} \to \mathfrak{Q}$. Because the germ $f_{iz} \in \mathfrak{B}$ depends continuously on $z \in U_i$, the section s is continuous. By definition and because of 2), $v(s) = f$.

b) Suppose that $f = v(s)$ with $s \in \Gamma(\bar{G}, \mathfrak{Q}) = \Gamma(\operatorname{supp}\mathfrak{Q}, \mathfrak{Q})$. For every $x \in \bar{G}$, an open neighborhood W_x of x in the space $\bar{G}$ and $f^x \in B_G(W_x)$ exists such that $\mu(f_z^x) = s(z)$ for all $z \in W_x$. If $W_x \cap W_y \neq \varnothing$ then $f_z^x - f_z^y \in \mathfrak{J}_z$ for all $z \in W_x \cap W_y$. Moreover,

$$f \,|\, W_x \cap S(\mathfrak{J}) = v(s) \,|\, W_x \cap S(\mathfrak{J}) = f^x \,|\, W_x \cap S(\mathfrak{J}). \qquad \text{Q.E.D.}$$

Lemma 2.7. *The holomorphic function $f \in A(G, S(\mathfrak{J}))$ is strictly bounded for $\mathfrak{J}$, if and only if an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of the space $\bar{G}$, a family $\{f_i\}_{i \in I}$ with $f_i \in B_G(U_i)$, and functions $h_{ij} \in B_G(U_i \cap U_j)$ for each $(i,j) \in I(1)$ exist such that:*

1. *If $i \in I$ and if $U_i \cap S(\mathfrak{J}) \neq \varnothing$, then $f \,|\, U_i \cap S(\mathfrak{J}) = f_i \,|\, U_i \cap S(\mathfrak{J})$.*
2. *If $(i,j) \in I(1)$, then $f_i - f_j = a_{ij} h_{ij}$ on $U_i \cap U_j \cap G$.*
3. *If $(i,j) \in I(1)$, then $\mathfrak{J} \,|\, U_i \cap U_j = h_{ij}\mathfrak{B} \,|\, U_i \cap U_j$.*

Proof. a) Obviously, if these conditions are satisfied, then f is strictly bounded.

b) Let f be strictly bounded. An open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of the space $\bar{G}$ and family $\{f_i\}_{i \in I}$ of functions $f_i \in B_G(U_i)$ exist such that the conditions of the definition of "strictly bounded" hold. These remain true for any refinement. Hence it can be assumed that $\mathfrak{J} \,|\, U_i$ is generated by an element of $B_G(U_i)$ such that there exists an open covering $\mathfrak{B} = \{V_i\}_{i \in I}$ of $\bar{G}$ with $\bar{V}_i$ compact and contained in U_i. Now, if $(i,j) \in I(1)$, an element $h_{ij} \in B_G(U_i \cap U_j)$ exists such that $\mathfrak{J} \,|\, U_i \cap U_j = h_{ij}\mathfrak{B} \,|\, U_i \cap U_j$. Then $(f_i - f_j)_z \in \mathfrak{J}_z$ for all $z \in U_i \cap U_j$. Lemma 2.3 gives $a_{ij} \in B_G(V_i \cap V_j)$ such that $f_i - f_j = a_{ij} h_{ij}$ on $V_i \cap V_j \cap G$. Hence the conditions 1–3 of the Lemma are satisfied for the covering $\mathfrak{B}$, the family $\{f_i \,|\, V_i \cap G\}_{i \in I}$ and the functions $h_{ij} \,|\, V_i \cap V_j \cap G$ and a_{ij} (observe that it was necessary to use Lemma 2.3). Q.E.D.

Now, the main result can be proved:

Theorem 2.8. *Let $G = G_1 \times \cdots \times G_n \neq \varnothing$ be an open, bounded and connected subset of $\mathbf{C}^n$ with $G_\nu \subset \mathbf{C}$ for $\nu = 1, \cdots, n$. Let $\mathfrak{B}$ be the sheaf of bounded holomorphic functions on $\bar{G}$. Let $\mathfrak{J}$ be a globally principal ideal sheaf in $\mathfrak{B}$ with $\mathfrak{J} \neq \mathfrak{B}$. Let f be a strictly bounded holomorphic function for $\mathfrak{J}$ on $S(\mathfrak{J})$. Then a bounded holomorphic function F on G exists such that*

$$F \mid S(\mathfrak{J}) = f.$$

Proof. The exact sequence $0 \to \mathfrak{J} \to \mathfrak{B} \to \mathfrak{Q} \to 0$ induces an exact sequence

$$0 \to \Gamma(\bar{G},\mathfrak{J}) \to \Gamma(\bar{G},\mathfrak{B}) \xrightarrow{\tilde{\mu}} \Gamma(\bar{G},\mathfrak{Q}) \to H^1(\bar{G},\mathfrak{J}) = 0.$$

Hence $\tilde{\mu}$ is surjective. Take $s \in \Gamma(\bar{G},\mathfrak{Q})$ such that $v(s) = f$. A bounded holomorphic function $F \in B_G(\bar{G}) = \Gamma(\bar{G},\mathfrak{B})$ exists such that $\tilde{\mu}(F) = s$. Hence $\mu(F_z) = s(z)$ for each $z \in \bar{G}$. Hence $f = v(s) = F \mid S(\mathfrak{J})$ by the construction of $v(s)$ using $W_x = \bar{G}$ and $f^x = F$ for all $x \in \bar{G}$. Q.E.D.

Theorem 2.8 and Stout's Theorem 2.4 imply:

Theorem 2.9. *Let $G = G_1 \times \cdots \times G_n$ where each $G_\nu = \{z \in \mathbf{C} \mid |z| < 1\}$ is the unit disc. Let $\mathfrak{B}$ be the sheaf of germs of bounded holomorphic functions on $\bar{G}$ and $\mathfrak{J}$ be a principal ideal sheaf in $\mathfrak{B}$ with $\mathfrak{J} \neq \mathfrak{B}$. Let f be a strictly bounded holomorphic function for $\mathfrak{J}$ on $S(\mathfrak{J})$. Then a bounded holomorphic function F exists on G such that $F \mid S(\mathfrak{J}) = f$.*

Define $G = \{(z,w) \in \mathbf{C}^2 \mid |z| < 1 \text{ and } |w| < 1\}$. Define $h(z,w) = (w - \frac{1}{2})(zw - \frac{1}{2})$. Define $\mathfrak{J} = h\mathfrak{B}$ on G. Define

$$S_1 = \{(z,\tfrac{1}{2}) \mid z \in \mathbf{C} \text{ with } |z| < 1\},$$
$$S_2 = \{(z,w) \in G \mid zw = \tfrac{1}{2}\}.$$

Then $S(\mathfrak{J}) = S_1 \cup S_2$. Define $f: S(\mathfrak{J}) \to \mathbf{C}$ by $f \mid S_1 = 0$ and $f \mid S_2 = 1$. Then f is bounded and holomorphic. Alexander [*1*] has shown that f is not the restriction of a bounded holomorphic function on G. Hence f is not strictly bounded!

NOTES

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2. See also Siu [5].

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